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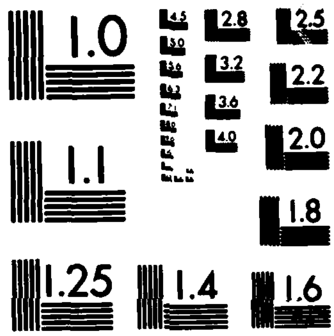

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REJECTION OF MULTIVARIATE OUTLIERS

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REJECTION OF MULTIVARIATE OUTLIERS

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Abstract

An extension of Ferguson's [Fourth Berkeley Symposium on Probability and Mathematical Statistics, 1961, Volume 1] univariate normal results for rejection of outliers is made to the multivariate case with mean slippage. The formulation is more general than that in Schwager and Margolin [Ann. Statist., 1982, Vol. 10, No. 3, 943-954] and the approach is also different. The main result can be viewed as a robustness property of Mardia's locally optimum multivariate normal kurtosis test to detect outliers against nonnormal multivariate distributions.

AMS 1980 Subject Classification: Primary 62A05, 62H15: Secondary 62H10, 62E15

Key words and phrases: Locally best invariant, maximal invariant, mean slippage, multivariate kurtosis, outliers, robustness, Wijsman's representation theorem.

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Rejection of Multivariate Outliers

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1. Introduction.

After the pioneering work of Ferguson (1961) on the detection and rejection of outliers in samples from a univariate normal distribution with either mean or variance slippage, recently a lot of work has been done on the problem of estimation of parameters and tests of hypotheses of parameters in some particular probability models, assuming the presence of some outliers in the data and appropriately taking care of this situation [Kale and Sinha (1971), Joshi (1972), Sinha (1972, 1973a,1973b,1973c,1975, 1976), Chikkagoudar and Kunchur (1980)]. These two aspects of the problem of outliers, as mentioned clearly in Schwager and Margolin (1982), are entirely different though. Generally, one needs one kind of techniques to determine if outliers are present and to identify them, and a different kind of techniques to suitably modify a statistical analysis for purposes of inferences to incorporate the information regarding the presence and identity of outlying observations.

The literature on these dual aspects of the outlier problem is vast. An excellent survey appears in Barnett and Lewis (1978) and also in Hawkins (1980). While most of the work in this direction is in the area of univariate distributions and that, too, often for a specific probability model, some aspects of the multivariate outlier problem are available in Siotani (1959), Karlin and Truax (1960), Healy (1968), Rohlf (1975) and also in Ferguson (1961). A discussion of multivariate outliers from a data analytic viewpoint is available in Gnanadesikan (1977).

The motivation for the present investigation lies in a recent paper of Schwager and Margolin (1982) who derive a locally optimum procedure for detection of multivariate normal outliers against departures in the mean. The paper has some very interesting features. First, after Ferguson's (1961) work, this seems to be the only paper attempting successfully to generalize the concept and techniques of Ferguson to the multinormal case. Secondly, interestingly enough, it turns out that the locally best invariant test for outliers (under a suitable group of transformations) against mean slippage alternatives is based on Mardia's (1970) multivariate sample kurtosis. However, the paper suffers from two drawbacks. It is argued that the multivariate normal error structure, the model actually adopted in the paper, has some advantages in terms of mathematical tractability etc. We will show that the assumption of multinormality of the error components can be dispensed with without essentially any difficulty. Another drawback, the less serious, is that the ultimate optimality result is obtained via some heavy calculations.

Our object in this paper is, therefore, two-fold. We extend the results of Schwager and Margolin (1982) to nonnormal multivariate populations and in the process provide a simpler derivation of the main optimality result. Interestingly enough, a new test also emerges out of this formulation (vide Remark 3.1), which is locally optimum uniformly for all mean slippage alternatives. This derivation of course depends mostly on the existing calculations of Schwager and Margolin (1982). Under a certain condition on the number of distinct outliers present in the data, an alternative independent proof of the main result is also provided. The class of nonnormal multivariate populations we consider is well known in the context of robust tests for some multivariate problems (see Kariya and Eaton (1977) Kariya (1981a,1981b),

sinha and Drygas (1982) for a univariate problem). The main tool we use is a representation theorem due to Wijsman (1967).

The multivariate outlier problem is formulated in Secion 2. Section 3 contains the main result. The alternative proof is provided in Section 4. Some of the calculations necessary for this section are deferred to in the Appendix.

2. The Multivariate Outlier Problem.

Let $\mathcal{O}(n)$ and $\mathcal{S}(p)$ denote the set of n×n orthogonal matrices and the set of p×p positive definite matrices respectively. For an n×p random matrix U, we denote by $\mathcal{L}(U)$ the distribution of U. We call U elliptically symmetric about M with scale matrix $\Sigma \in \mathcal{S}(p)$ if $\mathcal{L}(gY) = \mathcal{L}(Y)$ for all $g \in \mathcal{O}(np)$, where $Y = (\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n)'$, $\underset{\sim}{Y}_i$ is the ith row of $Y = (U-M)\Sigma^{-1/2}$. Let $\mathcal{X} = \{U\colon n\times p \mid \text{rank}\ (U) = p\}$. Throughout the paper, $n \geq p+1$ is assumed. Moreover, we denote by $\mathcal{F}(M, I_n \otimes \Sigma)$ the class of np-dimensional elliptically symmetric distributions about M with scale matrix $\Sigma \in \mathcal{S}(p)$ such that $P\{U-M \in \mathcal{X}\} \equiv 1$. We assume that $\mathcal{L}(U) \in \mathcal{F}(M, I_n \otimes \Sigma)$ has a density (with respect to the Lebesgue measure) which is expressible as

$$(2.1) \qquad f(U|M,\Sigma) = |\Sigma|^{-n/2}\, \phi(\text{tr}\ \Sigma^{-1}(U-M)'(U-M))$$

where $\phi\colon [0,\infty) \to [0,\infty)$.

Consider a random sample of size n from a multivariate distribution. We will denote the sample by X: n×p and assume that the following model holds:

$$(2.2) \qquad X = \underset{\sim}{1}\,\underset{\sim}{\mu}' + U\,\Sigma^{-1/2}$$

where $\underset{\sim}{1}$ is the unit vector (n×1), $\underset{\sim}{\mu}$ is the unknown common mean vector (p×1) of the rows of X and the random error component U has a distribution $\mathcal{L}(U) \in \mathcal{F}(0, I_n \otimes I_p)$

with a density given in (2.1) with $M = 0$ and $\Sigma = I_p$. This is equivalent to the specification that $L(X) \in F(\underset{\sim}{1}\underset{\sim}{\mu}', I_n \otimes \Sigma)$. It is clear that our formulation is more general than Schwager and Margolin's (1982) in that ϕ is arbitrary. Some mild regularity conditions on ϕ which will be needed in the sequel are presented later.

In this paper we consider the possibility of outliers with mean slippage. For any matrix $A = (a_{ij})$: $n \times p$, extending Ferguson's (1961) formulation and proceeding as in Schwager and Margolin (1982), this can be incorporated by considering the model

$$X = \underset{\sim}{1}\underset{\sim}{\mu}' + \Delta A \Sigma^{-1/2} + U \Sigma^{-1/2} \tag{2.3}$$

Here Δ is a nonnegative scalar and A is an arbitrary matrix such that some of the rows of A are zero. In this formulation, unless $\Delta = 0$, the observation X_i corresponding to the ith row of X is an outlier if the ith row of A is nonzero. In Section 4 we assume that the rank of A is p which imposes a constraint on the number of distinct outliers present in the data whenever $\Delta > 0$. When n is large compared to p, such a condition may not be unrealistic. We remark that Schwager and Margolin (1982) assume on the other hand that more than half of the rows of A are zero and for large n this still might ensure rank (A) equal to p.

Henceforth, we will assume that we have chosen an arbitrary matrix A whose some of the rows are zero. The general multivariate outlier problem then consists of the model (2.3), the distributional assumption about U: $L(U) \in F(0, I_n \otimes I_p)$, and the null hypothesis H_0: $\Delta = 0$ versus the alternative hypothesis H_1: $\Delta > 0$. In what follows we will derive a locally optimum test of H_0 against H_1 using invariance principle. It may be noted that rejection of H_0 implies presence of

outliers in the data according to the assumed structure of A. However, as will be seen later (vide Remark 3.2), the optimum test obtains for a wide variety of outlier structures.

Following Schwager and Margolin (1982), it is clear that the above testing problem remains invariant under the action of the group $G = \mathcal{P} \times Gl(p) \times \mathbb{R}^p$ where $\mathcal{P}$ denotes the group of all n×n permutation matrices with elements Γ_α, Gl(p) the group of p×p nonsingular matrices with elements C and $\mathbb{R}^p$ the Euclidean p-space. The three (sub)group operations are defined by: (1) addition of an arbitrary vector $\mu^* \in \mathbb{R}^p$ to each row of X; (2) postmultiplication of X by any nonsingular matrix $C \in Gl(p)$ and (3) permutation of the rows of X by premultiplying X by $\Gamma_\alpha \in \mathcal{P}$. For details see Schwager and Margolin (1982). In the next section we derive the distribution of a maximal invariant statistic, applying Wijsman's (1967) theorem. This method does not require an explicit evaluation of a maximal invariant statistic although this is available in Schwager and Margolin (1982).

3. Main Results.

Without loss of generality, by invariance of the problem, we may assume $\underset{\sim}{\mu} = \underset{\sim}{0}$ and $\Sigma = I_p$. To obtain a formal expression of the distribution of a maximal invariant T(X), we use the following version of Wijsman's (1967) theorem.

Lemma 3.1. Let $h(x|\Delta) = \phi\,(\mathrm{tr}(x-\Delta A)'(x-\Delta A))$ be the pdf of X, let $T = t(X)$ be a maximal invariant under the transformation G and let P_Δ^T be the distribution induced by T under Δ. Then the pdf of T with respect to P_0^T evaluated at $T = t(X)$ is given by

$$(3.1)\qquad \frac{dP_\Delta^T}{dP_0^T}(t(X)) = \frac{\int_G h(g\cdot X|\Delta)\,|C'C|^{n/2}\,d\nu(g)}{\int_G h(g\cdot X|\Delta=0)\,|C'C|^{n/2}\,d\nu(g)}$$

where ν is a left invariant measure on G.

Conditions under which (3.1) holds are stated in Wijsman (1967) and have been verified in many contexts under similar group structures (see Kariya and Eaton (1977), Kariya (1978, 1981a, 1981b). The details are omitted here. For some applications of Wijsman's (1967) theorem in multivariate testing problems with incomplete data see Eaton and Kariya (1975) and Sinha et al (1982).

We now proceed to explicitly evaluate (3.1). The transformation $g\cdot x$ is given by (vide Schwager and Margolin (1982))

$$(3.2)\qquad g\cdot x = \Gamma_\alpha x C + \underset{\sim}{1}\,\underset{\sim}{\mu}^{*'}, \; \Gamma_\alpha \in P, \; C \in Gl(p), \; \underset{\sim}{\mu}^* \in \mathbb{R}^p$$

and we take $\nu = \nu_1 \times \nu_2 \times \nu_3$ where ν_1 is the discrete uniform probability measure with mass $\frac{1}{n!}$ at each of the n! elements $\Gamma_\alpha \in P$, $d\nu_2(C) = dC/|C'C|^{p/2}$ and $d\nu_3(\underset{\sim}{\mu}^*)$ is the Lebesgue measure on $\mathbb{R}^p$.

Lemma 3.2. The ratio of the pdfs in expression (3.1) is evaluated as

$$(3.3)\qquad \frac{\Sigma_\alpha \int_{Gl(p)} \tilde{\phi}[tr(C'C-2\Delta C'S^{-1/2}(\Gamma_\alpha x-\underset{\sim}{1}\bar{\underset{\sim}{x}}')'A+\Delta^2(A'A - \frac{A'\underset{\sim}{1}\underset{\sim}{1}'A}{n})]|C'C|^{\frac{n-p}{2}}dC}{\Sigma_\alpha \int_{Gl(p)} \tilde{\phi}[tr(C'C)]|C'C|^{\frac{n-p}{2}}dC}$$

for some $\tilde{\phi} : [0,\infty) \to [0,\infty)$, where Σ_α denotes the summation over n! terms representing permutations of the rows of x, $\bar{\underset{\sim}{x}}$ the sample mean vector and $S = x'x - n\,\bar{\underset{\sim}{x}}\bar{\underset{\sim}{x}}'$.

Proof. The numerator N_Δ (say) of (3.1) can be written as

$$(3.4)\qquad N_\Delta = \frac{1}{n!}\Sigma_\alpha \int_{Gl(p)} \int_{\mathbb{R}^p} \phi(tr(x'x-\Delta x'\,A-\Delta A'x+\Delta^2 A'A|x\to g\cdot x))|C'C|^{\frac{n-p}{2}}dC\,d\nu_3(\underset{\sim}{\mu}^*).$$

The argument of ϕ, after the substitution $x = g\cdot x$, simplifies to

$$(3.5)\quad \mathrm{tr}[(\Gamma_\alpha xC+\underset{\sim}{1}\,\underset{\sim}{\mu}^{*'})'(\Gamma_\alpha xC+\underset{\sim}{1}\,\underset{\sim}{\mu}^{*'}) - 2\Delta A'(\Gamma_\alpha xC+\underset{\sim}{1}\,\underset{\sim}{\mu}^{*'}) + \Delta^2 A'A]$$

$$= \mathrm{tr}(C'x'\Gamma_\alpha'\Gamma_\alpha xC + 2n\, C'\underset{\sim\sim}{\bar{x}\mu}^{*'} + n\,\underset{\sim}{\mu}^*\underset{\sim}{\mu}^{*'} - 2\Delta A'\Gamma_\alpha xC - 2\Delta A'\underset{\sim\sim}{1\mu}^{*'} + \Delta^2 A'A)$$

$$= \mathrm{tr}(C'x'xC - 2\Delta A\Gamma_\alpha xC + \Delta^2 A'A - n(C'\underset{\sim}{\bar{x}} - \frac{\Delta}{n}A'\underset{\sim}{1})(\underset{\sim}{\bar{x}}'C - \frac{\Delta}{n}\underset{\sim}{1}'A) + n\,\underset{\sim}{c}^*\underset{\sim}{c}^{*'})$$

$$= \mathrm{tr}(C'(x'x-n\underset{\sim\sim}{\bar{x}\bar{x}}')C - 2\Delta C'(\Gamma_\alpha x - \underset{\sim\sim}{1\bar{x}}')'A + \Delta^2(A'A - \frac{A'\underset{\sim\sim}{11}'A}{n}) + n\,\underset{\sim}{c}^*\underset{\sim}{c}^{*'}).$$

In the equalities above, $\underset{\sim}{c}^* = \underset{\sim}{\mu}^* - \frac{\Delta}{n}\underset{\sim}{1}'A + \underset{\sim}{\bar{x}}'C$ and we have used the fact that $x'\Gamma_\alpha'\Gamma_\alpha x = x'x,\ \forall\ \Gamma_\alpha \in P$. Since $d\nu_3(\underset{\sim}{\mu}^*) = d\nu_3(\underset{\sim}{c}^*)$, using a result of Dawid (1977), integration with respect to $\underset{\sim}{c}^*$ over $\mathbb{R}^p$ yields

$$(3.6)\quad N_\Delta = \frac{1}{n!}\Sigma_\alpha \int_{Gl(p)} \tilde{\phi}[\mathrm{tr}(C'(x'x-n\underset{\sim\sim}{\bar{x}\bar{x}}')C - 2\Delta C'(\Gamma_\alpha x - \underset{\sim\sim}{1\bar{x}}')'A + \Delta^2(A'A - \frac{A'\underset{\sim\sim}{11}'A}{n}))]\,|C'C|^{\frac{n-p}{2}} dC$$

for some $\tilde{\phi}:[0,\infty) \to [0,\infty)$.

Now $x'x - n\underset{\sim\sim}{\bar{x}\bar{x}}' = S$, the sample sum of squares and products matrix, is p.d. by our assumption $n \geq p+1$. Writing $S = S^{1/2}S^{1/2}$ where $S^{1/2}$ is the positive square root of S and making the transformation $C \to S^{1/2}C$, N_Δ reduces to

$$(3.7)\quad N_\Delta = \frac{|S|^{-n/2}}{n!}\Sigma_\alpha \int_{Gl(p)} \tilde{\phi}[\mathrm{tr}(C'C - 2\Delta C'S^{-1/2}(\Gamma_\alpha x - \underset{\sim\sim}{1\bar{x}}')'A + \Delta^2(A'A - \frac{A'\underset{\sim\sim}{11}'A}{n}))]\,|C'C|^{\frac{n-p}{2}} dC$$

Since the denominator of (3.1) corresponds to N_Δ with $\Delta = 0$, the lemma follows.

□

We now proceed to explicitly evaluate the expression in (3.3). Making a transformation from C to −C, it is clear from (3.3) that the ratio of the pdfs depends only on Δ^2. We show that the coefficient of Δ^2 in this ratio

$$(3.8)\quad \delta_2 = \Sigma_\alpha \int_{Gl(p)} [\mathrm{tr}\{AC'S^{-1/2}(\Gamma_\alpha x - \underset{\sim\sim}{1\bar{x}}')'\}]^2\ \tilde{\phi}^{(2)}(\mathrm{tr}\ C'C)\,|C'C|^{\frac{n-p}{2}} dC,$$

apart from another constant term depending only on A, is a constant, and that the coefficient of Δ^4 in the ratio

$$(3.9)\quad \delta_4 = \Sigma_\alpha \int_{Gl(p)} [\mathrm{tr}\{AC'S^{-1/2}(\Gamma_\alpha x-\underset{\sim}{1}\bar{\underset{\sim}{x}}')'\}]^4\ \tilde{\phi}^{(4)}(\mathrm{tr}\ C'C)\ |C'C|^{\frac{n-p}{2}}\ dC,$$

apart from other terms including (3.8) which are constants, is of the form $K_1T(x)\cdot L(A)+K_2$ where $K_1>0$, K_2 are constants,

$$(3.10)\qquad T(x) = b_{2,p} \equiv n\sum_1^n \{(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})\}^2$$

and

$$(3.11)\qquad L(A) = (n^3+n^2)\sum_{i=1}^n \|\gamma_i\|^4 - (n^2-n)[2\sum_{i,j=1}^n (\gamma_i'\gamma_j)^2 + (\sum_{i=1}^n \|\gamma_i\|^2)^2],$$

with $\underset{\sim}{\gamma}_i = (\underset{\sim}{a}_i-\bar{\underset{\sim}{a}})$, $\|\underset{\sim}{\gamma}_i\|^2 = \underset{\sim}{\gamma}_i'\underset{\sim}{\gamma}_i$, $x' = (\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n)$ and $A' = (\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_n)$.

<u>Evaluation of δ_2</u>. This is based on the following elementary result whose proof is omitted.

<u>Lemma 3.3</u>. $\Sigma_\alpha\{\mathrm{tr}(BA_\alpha)\}^2 = \underset{\sim}{1}'B(\Sigma_\alpha A_\alpha A_\alpha')B'\ \underset{\sim}{1}$.

Taking $B = AC'$, $A_\alpha = S^{-1/2}(\Gamma_\alpha x-\underset{\sim}{1}\bar{\underset{\sim}{x}}')'$ and using the fact that $\Sigma_\alpha(\Gamma_\alpha x-\underset{\sim}{1}\bar{\underset{\sim}{x}}')'(\Gamma_\alpha x-\underset{\sim}{1}\underset{\sim}{x}')/n!$ $= S$, it follows that

$$(3.12)\qquad \delta_2 = n!\int_{Gl(p)} (\underset{\sim}{1}'AC'CA'\underset{\sim}{1})\ \tilde{\phi}^{(2)}(\mathrm{tr}\ C'C)|C'C|^{\frac{n-p}{2}}dC$$

which is a constant.

<u>Evaluation of δ_4</u>. This is primarily based on some results derived by Schwager and Margolin (1982). Write $\mathrm{tr}\{AC'S^{-1/2}(\Gamma_\alpha x-\underset{\sim}{1}\bar{\underset{\sim}{x}}')'\} = \mathrm{tr}\{\Gamma_\alpha' AC'S^{-1/2}(x-\underset{\sim}{1}\bar{\underset{\sim}{x}}')'\} = \sum_{i=1}^n \underset{\sim}{\xi}_i'\ C\ \underset{\sim}{a}_{i\alpha}$ where x is any specified realization of $\Gamma_\alpha x$, $\underset{\sim}{\xi}_i = S^{-1/2}\underset{\sim}{\delta}_i$, $\underset{\sim}{\delta}_i$ is the ith column vector of $(x-\underset{\sim}{1}\bar{\underset{\sim}{x}}')'$ and $\underset{\sim}{a}_{i\alpha}$ is the ith column vector of $A'\Gamma_\alpha$. Since $\sum_{i=1}^n \underset{\sim}{\xi}_i$ is a null vector, in the last equality above we can replace $\underset{\sim}{a}_{i\alpha}$ by $\underset{\sim}{\gamma}_{i\alpha}$ =

the ith column vector of $(A'\Gamma_\alpha - \bar{a}1')$. We now use the results of Section 5 of Schwager and Margolin (1982). Their Theorem 5.1 is applicable directly and their Lemmas 5.1 and 5.2 and Theorem 5.2 are used with Φ suitably redefined as

$$\Phi = \int_{Gl(p)} c_{11}^4 \, \tilde{\phi}^{(4)}(\mathrm{tr}\, C'C) |C'C|^{\frac{n-p}{2}} dC.$$

The justification follows easily because the above results are independent of any particular structure of the underlying probability distribution and actually depend on the invariance of the associated measure under orthogonal transformations. This yields

$$\delta_4 = (n-4)! \, \Phi \, L(A) \, \{ \sum_{i=1}^{n} ||\xi_i||^4 \} + K \tag{3.13}$$

where L(A) is as defined in (3.11). This completes our demonstration since $||\xi_i||^2 = (x_i - \bar{x})' S^{-1} (x_i - \bar{x})$.

Incidentally, our argument leading to the evaluation of δ_4 implies that L(A) can also be defined in terms of a_i's instead of γ_i's. Our main result is the following.

<u>Theorem 3.1</u>. Assume that $\tilde{\phi}$ satisfies (i) $\Phi < \infty$, (ii) $\tilde{\phi}(x+y)$ admits a Taylor expansion in y for every x with continuous fourth order derivative, and (iii) that the first four derivatives of the power function of an invariant test with respect to Δ can be carried out beneath the integral sign. The locally best invariant locally unbiased test, conditionally on A, rejects H_0: $\Delta = 0$ for large values of $b_{2,p}$ if $\Phi \cdot L(A) > 0$ and small values of $b_{2,p}$ if $\Phi \cdot L(A) < 0$.

<u>Proof</u>. An application of Lemma 3.2 and the generalized Neyman-Pearson Lemma along with the quantities δ_2, δ_4 completes the proof of the theorem. The routine details are omitted (see, for example, Kariya (1981b)). □

Remark 3.1. Suppose for a moment that we evaluate δ_4 the other way around. The argument used to evaluate δ_4 along with the symmetry of x and A then yields another expression for δ_4 given by

$$\delta_4 = (n-4)!\ \Phi\ L(x)\{\sum_{i=1}^{n}\|\gamma_i\|^4\} + K \tag{3.14}$$

where $\sum_{i=1}^{n}\|\gamma_i\|^4 = \sum_{i=1}^{n}\{(\underset{\sim}{a}_i-\bar{\underset{\sim}{a}})'(\underset{\sim}{a}_i-\bar{\underset{\sim}{a}})\}^2$ and L(x) has the same expression as L(A) in (3.11) with γ_i replaced by $\underset{\sim}{\xi}_i$, $i = 1,\ldots,n$. i.e. $L(x) = (n^3+n^2)\sum_{i=1}^{n}\{(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})\}^2 - (n^2-n)[2\sum_{i,j=1}^{n}\{(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}_j-\bar{\underset{\sim}{x}})\}^2 + \{\sum_{i=1}^{n}(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})'S^{-1}(\underset{\sim}{x}_i-\bar{\underset{\sim}{x}})\}^2]$. It then follows that, unless $\underset{\sim}{a}_i$'s are equal, the locally best invariant locally unbiased test rejects H_0 for large values of L(x) whenever $\phi > 0$. This, to our knowledge, is a new test statistic. Interestingly though, for $p = 1$, this boils down to the familiar univariate b_2-test (Ferguson, 1961).

Remark 3.2. As mentioned in Schwager and Margolin (1982), the local optimality of the $b_{2,p}$-test obtains for a specific A and a universal optimality (local) result holds whenever the fraction of nonzero rows of A is at most 21%. On the other hand, it follows from Remark 3.1 that whenever there are some outliers, the test based on $\phi L(x)$ is locally optimum.

Remark 3.3. The class of nonnormal distributions ϕ considered in (2.1) contains the (np-dimensional) multivariate t-distribution, the multivariate Cauchy distribution, the contaminated normal distribution and more generally continuous normal mixtures of the type $f = \int_0^\infty N(U|M,a\Sigma)\ dG(a)$ where N is the normal density and G is a distribution function on $(0,\infty)$. For such an f, it is easy to justify the conditions in Theorem 3.1 (see Ferguson (1961)) and it turns out that $\Phi > 0$.

4. Alternative Proof.

In this section we provide an alternative proof of the main result under the assumption that the rank of A is p. The key to this is the following.

Lemma 4.1. Under the assumption rank $(A) = p$, the ratio of the pdfs in (3.3) further reduces to

$$\frac{\Sigma_\alpha \int_{Gl(p)} \tilde{\phi}[\operatorname{tr} C'C - 2\Delta \operatorname{tr} D_\alpha^{1/2} C + \Delta^2 \operatorname{tr}(A'A - \frac{A'\underset{\sim}{1}\underset{\sim}{1}'A}{n})]\,|C'C|^{\frac{n-p}{2}}\, dC}{\Sigma_\alpha \int_{Gl(p)} \tilde{\phi}[\operatorname{tr} C'C]\ |C'C|^{\frac{n-p}{2}}\, dC} \tag{4.1}$$

where D_α is a diagonal matrix with the diagonal elements as the eigen values of the matrix $A'(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')\ S^{-1}(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')'A$.

Proof. Clearly rank $(A) = p$ implies that the matrix $(\Gamma_\alpha x - \underset{\sim}{1}\bar{x})'A$ is nonsingular for almost all x and therefore the matrix P defined by

$$P = [A'(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')S^{-1}(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')'A]^{-1/2}\ A'(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')S^{-1/2} \tag{4.2}$$

is orthogonal. The transformation $C \to PC$ then reduces N_Δ displayed in (3.7) to

$$N_\Delta = \frac{|S|^{-n/2}}{n!}\Sigma_\alpha \int_{Gl(p)} \tilde{\phi}[\operatorname{tr}(C'C - 2\Delta(A'(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')S^{-1}(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')'A)^{1/2}C + \Delta^2(A'A - \frac{A'\underset{\sim}{1}\underset{\sim}{1}'A}{n}))] \cdot |C'C|^{\frac{n-p}{2}}\, dC \tag{4.3}$$

Write $A'(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')S^{-1}(\Gamma_\alpha x - \underset{\sim}{1}\bar{x}')'A = Q_\alpha' D_\alpha Q_\alpha$ for some orthogonal Q_α and diagonal $D_\alpha = \operatorname{diag}(\lambda_{1\alpha},\ldots,\lambda_{p\alpha})$ where $\lambda_{1\alpha},\ldots,\lambda_{p\alpha}$ are the eigen values of the associated matrix. The measure $dC/|C'C|^{p/2}$ being both left and right invariant, it follows that a double substitution $C \to Q_\alpha C \to CQ_\alpha'$ in (4.3) transforms N_Δ to the final form

$$(4.4)\quad N_\Delta = \frac{|S|^{-n/2}}{n!}\,\Sigma_\alpha \int_{Gl(p)} \tilde{\phi}[\mathrm{tr}\, C'C - 2\Delta\, \mathrm{tr}\, D_\alpha^{1/2}C + \Delta^2 \mathrm{tr}(A'A - \frac{A'\underset{\sim}{1}\underset{\sim}{1}'A}{n})]\, |C'C|^{\frac{n-p}{2}} dC,$$

thereby proving the lemma. □

The appropriate version of the main result is stated below.

Theorem 4.1. Assume that $\tilde{\phi}$ satisfies the conditions of Theorem 3.1. The locally best invariant locally unbiased test, conditionally on A, rejects H_0: $\Delta = 0$ for large values of $b_{2,p}$ if $\Phi L^*(A) > 0$ and small values of $b_{2,p}$ if $\Phi L^*(A) < 0$, where Φ is as defined in Section 3 and $L^*(A)$ is defined in (4.7).

Proof. There are two main steps in the proof of this theorem. As in Section 3, making a transformation from C to -C, it is clear from (4.1) that the ratio of the pdfs depends only on Δ^2. We show that the coefficient of Δ^2 in this ratio

$$(4.5)\qquad \Sigma_\alpha \int_{Gl(p)} \{\mathrm{tr}\, D_\alpha^{1/2}C\}^2\, \tilde{\phi}^{(2)}(\mathrm{tr}\, C'C)\,|C'C|^{\frac{n-p}{2}} dC,$$

apart from another constant term depending only on A, is a constant, and that the coefficient of Δ^4 in the ratio

$$(4.6)\qquad \Sigma_\alpha \int_{Gl(p)} \{\mathrm{tr}\, D_\alpha^{1/2}C\}^4\, \tilde{\phi}^{(4)}(\mathrm{tr}\, C'C)\,|C'C|^{\frac{n-p}{2}} dC,$$

apart from other terms including (4.5) which are constants, is of the form $K_1 b_{2,p}\Phi L^*(A)+K$ where $K_1 > 0$, K_2 are constants and

$$(4.7)\quad L^*(A) = (n^3+n^2-4n)\sum_{k=1}^{n}(\sum_{i=1}^{p} a_{ik}^2)^2 - (2n^2-4n)\sum_{i=1}^{p}\sum_{j=1}^{p}(\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_j)^2 - (n^2+n)\{\sum_{i=1}^{p}(\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_i)\}^2$$

$$+ 8n\sum_{i=1}^{p}\sum_{j=1}^{p}(\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_j)(\underset{\sim}{\alpha}_i'\underset{\sim}{1})(\underset{\sim}{\alpha}_j'\underset{\sim}{1}) + 4n(\sum_{i=1}^{p}\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_i)\{\sum_{i=1}^{p}(\underset{\sim}{\alpha}_i'\underset{\sim}{1})^2\}$$

$$- 4(n^2+n)\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{n}(\underset{\sim}{\alpha}_i'\underset{\sim}{1})\, a_{ik}a_{jk}^2 - 6\{\sum_{i=1}^{p}(\underset{\sim}{\alpha}_i'\underset{\sim}{1})^2\}^2$$

where $\underset{\sim}{\alpha}_i$ is the ith column of A.

An application of the generalized Neyman-Pearson lemma then completes the proof of the theorem. □

<u>Evaluations of (4.5) and (4.6)</u>. We use the following general result. For any integrable function $\psi(\mathrm{tr}\, C'C) : [0,\infty) \to [0,\infty)$,

$$(4.8) \qquad \int_{Gl(p)} c_{ii}^{\nu_1} c_{jj}^{\nu_2} c_{\ell\ell}^{\nu_3} \psi(\mathrm{tr}\, C'C) |C'C|^{\frac{n-p}{2}} dC = \begin{cases} \delta^{(\psi)}_{\nu_1+\nu_2+\nu_3} & \text{for } i=j=\ell,\ \nu_1+\nu_2+\nu_3 \text{ even} \\ \delta^{(\psi)}_{\nu_1+\nu_2,\nu_3} & \text{for } i=j\neq\ell,\ \nu_1+\nu_2 \text{ and } \nu_3 \text{ even} \\ \left[(\delta^{(\psi)}_{\nu_3,\nu_1+\nu_2}) \right] & \\ 0 & \text{for } i\neq j\neq\ell, \text{ at least one of } \nu_1,\nu_2\ \nu_3 \text{ odd} \\ 0 & \text{for } i=j\neq\ell, \text{ at least one of } \nu_1+\nu_2 \text{ and } \nu_3 \text{ odd} \end{cases}$$

where δ's are constants. Using (4.8), the expression in (4.5) reduces to

$$(4.9) \qquad \Sigma_\alpha \Big(\sum_{i=1}^{p} \lambda_{i\alpha}\Big) \cdot \delta_2(\tilde{\phi}^{(2)})$$

and that in (4.6) to

$$(4.10) \qquad \Sigma_\alpha \Big(\sum_{i=1}^{p} \lambda_{i\alpha}^2\Big) \delta_4(\tilde{\phi}^{(4)}) + 3\, \Sigma_\alpha \Big(\sum_{i\neq j} \lambda_{i\alpha}\lambda_{j\alpha}\Big) \delta_{2,2}(\tilde{\phi}^{(4)}).$$

<u>Lemma 4.2</u>. $\delta_{2,2}(\tilde{\phi}^{(4)}) = \frac{1}{3}\, \delta_4(\tilde{\phi}^{(4)})$.

<u>Proof</u>. The proof of this result follows from that of Lemma 5.2 of Schwager and Margolin (1982). Note that this latter result proved under the normality assumption also holds whenever the underlying distribution is orthogonally invariant. □

In view of Lemma 4.2, the expression in (4.10) boils down to

$$(4.11) \qquad \Sigma_\alpha \Big(\sum_{i=1}^{p} \lambda_{i\alpha}\Big)^2 \cdot \delta_4(\tilde{\phi}^{(4)})$$

Note that $\delta_4(\tilde{\phi}^{(4)})$ is nothing but Φ defined in Section 3. We now prove the following result towards evaluating (4.9) and (4.11).

Lemma 4.3. (a) $\Sigma_\alpha(\sum_{i=1}^{p} \lambda_{i\alpha})$ = a constant (depending on A)

(b) $\Sigma_\alpha(\sum_{i=1}^{p} \lambda_{i\alpha})^2 = \frac{b_{2,p}}{n}(n-4)!L^*(A)$ + a constant where $L^*(A) \equiv \sum_i\sum_j \underset{\sim}{\alpha}_i'((\delta_{i,j,k,k'}))\underset{\sim}{\alpha}_j$ is evaluated in (4.22).

Proof. (a)

$$(4.12) \qquad \sum_{i=1}^{p} \lambda_{i\alpha} = \mathrm{tr}\, D_\alpha = \mathrm{tr}\, A'(\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')S^{-1}(\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')'A$$

$$= \sum_{i=1}^{p} \underset{\sim}{e}_i'A'(\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')S^{-1}(\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')'A\,\underset{\sim}{e}_i$$

$$= \sum_{i=1}^{p} \underset{\sim}{\alpha}_i'(\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')S^{-1}(\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')'\underset{\sim}{\alpha}_i$$

$$= \sum_{\ell=1}^{p} \sum_{i=1}^{p} (\underset{\sim}{\alpha}_i'\underset{\sim}{z}_{\ell\alpha})^2$$

where $\underset{\sim}{e}_i' = (0\ldots1\ldots0)$ with 1 at the ith coordinate, $\underset{\sim}{\alpha}_i$ = the ith column of $A(n\times1)$, S^{-1} is written as $S^{-1} = \sum_{\ell=1}^{p} \frac{1}{\lambda_\ell} \underset{\sim}{P}_\ell \underset{\sim}{P}_\ell'$ and $\underset{\sim}{Z}_{\ell\alpha} = (\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')\underset{\sim}{P}_\ell/\sqrt{\lambda_\ell}$. Hence, using a result of Ferguson (1961),

$$(4.13) \qquad \Sigma_\alpha(\sum_{i=1}^{p} \lambda_{i\alpha}) = \sum_{\ell=1}^{p} \sum_{i=1}^{p} \{\Sigma_\alpha(\underset{\sim}{\alpha}_i'\underset{\sim}{z}_{\ell\alpha})^2\} = \sum_{\ell=1}^{p} \sum_{i=1}^{p} (n-2)![n\,(\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_i) - (\underset{\sim}{\alpha}_i'\underset{\sim}{1})^2]\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{z}_{\ell\alpha}$$

$$= (n-2)![\sum_{i=1}^{p} \{n(\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_i) - (\underset{\sim}{\alpha}_i'\underset{\sim}{1})^2\}][\sum_{\ell=1}^{p} \{\underset{\sim}{P}_\ell'(\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')'(\Gamma_\alpha \underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')\underset{\sim}{P}_\ell/\lambda_\ell\}]$$

$$= (n-2)![\sum_{i=1}^{p} \{n(\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_i) - (\underset{\sim}{\alpha}_i'\underset{\sim}{1})^2\}][\sum_{\ell=1}^{p} \underset{\sim}{P}_\ell'S\underset{\sim}{P}_\ell/\lambda_\ell] = p(n-2)![\sum_{i=1}^{p} \{n(\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_i) - (\underset{\sim}{\alpha}_i'\underset{\sim}{1})^2\}$$

thereby proving the first part of the lemma.

(b) The proof for this part of the lemma is slightly involved. Using the last but one step of (4.12), we can write

$$(4.14)\quad \left(\sum_{i=1}^{p}\lambda_{i\alpha}\right)^2 = \sum_{j=1}^{p}\sum_{i=1}^{p}\underset{\sim}{\alpha}_i'(\Gamma_\alpha \underset{\sim}{x}-\underset{\sim}{1}\bar{\underset{\sim}{x}}')S^{-1}(\Gamma_\alpha \underset{\sim}{x}-\underset{\sim}{1}\bar{\underset{\sim}{x}}')'\underset{\sim}{\alpha}_i\cdot\underset{\sim}{\alpha}_j'(\Gamma_\alpha \underset{\sim}{x}-\underset{\sim}{1}\bar{\underset{\sim}{x}}')S^{-1}(\Gamma_\alpha \underset{\sim}{x}-\underset{\sim}{1}\underset{\sim}{x}')'\underset{\sim}{\alpha}_j$$

$$= \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{j=1}^{p}\sum_{i=1}^{p}(\underset{\sim}{\alpha}_i'\underset{\sim}{z}_{\ell\alpha})^2(\underset{\sim}{\alpha}_j'\underset{\sim}{z}_{\ell'\alpha})^2$$

$$= \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{j=1}^{p}\sum_{i=1}^{p}\underset{\sim}{\alpha}_i'(\underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}')\underset{\sim}{\alpha}_i\ \underset{\sim}{\alpha}_j'(\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}')\ \underset{\sim}{\alpha}_j\ .$$

The $n\times n$ matrix $(\underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\alpha\ell}')\ \underset{\sim}{\alpha}_i\underset{\sim}{\alpha}_j'(\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}') = \Delta_{\ell,\ell',i,j,\alpha}$ (say) appearing in (4.14) can be expressed as the product of three matrices of orders $n\times n^2$, $n^2\times n^2$ and $n^2\times n$ respectively as shown below.

$$(4.15)\quad \Delta_{\ell,\ell',i,j,\alpha} = \begin{pmatrix}\underset{\sim}{\alpha}_i' & \underset{\sim}{0} & \cdots & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{\alpha}_i' & \cdots & \underset{\sim}{0}\\ \cdot & \cdot & \cdots & \cdot\\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{\alpha}_i'\end{pmatrix}\begin{pmatrix}\underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{e}_1\underset{\sim}{e}_1'\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}' & \cdots & \underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{e}_1\underset{\sim}{e}_n'\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}'\\ \cdot & \cdots & \cdot\\ \underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{e}_i\underset{\sim}{e}_1'\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}' & \cdots & \underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{e}_i\underset{\sim}{e}_n'\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}'\\ \cdot & \cdots & \cdot\\ \underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{e}_n\underset{\sim}{e}_1'\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}' & \cdots & \underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{e}_n\underset{\sim}{e}_n'\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}'\end{pmatrix}\ .$$

$$\begin{pmatrix}\underset{\sim}{\alpha}_j & \underset{\sim}{0} & \cdots & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{\alpha}_j & \cdots & \underset{\sim}{0}\\ \cdot & \cdot & \cdots & \cdot\\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{\alpha}_j\end{pmatrix}$$

The crux of the problem now is to evaluate the sum of all possible $n^2\times n^2$ matrices appearing in the middle of (4.15) over ℓ,ℓ' and permutations α. This is carried out in the Appendix. It is proved that

$$(4.16)\quad \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\Sigma_\alpha \underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{e}_k\underset{\sim}{e}_k'\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}' = \frac{b_{2,p}}{n}\,(n-3)!\,[2\ \underset{\sim}{1}\underset{\sim}{1}' + (n^2-2n)\underset{\sim}{e}_k\underset{\sim}{e}_k' - n\Delta_k]$$

$$+ \text{ another matrix of constants}$$

$$(4.17)\quad \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\Sigma_\alpha \underset{\sim}{z}_{\ell\alpha}\underset{\sim}{z}_{\ell\alpha}'\underset{\sim}{e}_k\underset{\sim}{e}_{k'}'\underset{\sim}{z}_{\ell'\alpha}\underset{\sim}{z}_{\ell'\alpha}' = \frac{b_{2,p}}{n}\,(n-4)!\,[-6\ \underset{\sim}{1}\underset{\sim}{1}' + (-n^2+3n)(\underset{\sim}{e}_k\underset{\sim}{e}_k' + \underset{\sim}{e}_{k'}\underset{\sim}{e}_{k'}' + \underset{\sim}{e}_k\underset{\sim}{e}_{k'}'$$

$$+ \underset{\sim}{e}_{k'}\underset{\sim}{e}_k') + 2n\ \Delta_{k.k'}]$$

$$+ \text{ a matrix of constants}$$

where Δ_k is a matrix with 1's in the diagonal and also in the kth row and kth column and 0 elsewhere, and $\Delta_{k,k'}$ is a matrix with 1's in the diagonal and also in the kth row, k'th row, kth column, k'th column and 0 elsewhere. Using (4.16) and (4.17), we get

$$(4.18)\quad \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p} \Sigma_\alpha \Delta_{\ell,\ell',i,j,\alpha} = \frac{b_{2,p}}{n}(n-4)!((\delta_{i,j,k,k'})) + \text{a matrix of constants}$$

where

$$(4.19)\quad \delta_{i,j,k,k} = 2(n-3)(\boldsymbol{\alpha}_i'\mathbf{1})(\boldsymbol{\alpha}_j'\mathbf{1})+(n-3)(n^2-2n)(\boldsymbol{\alpha}_i'\mathbf{e}_k)(\boldsymbol{\alpha}_j'\mathbf{e}_k)-n(n-3)\boldsymbol{\alpha}_i'\Delta_k\boldsymbol{\alpha}_j$$

$$\delta_{i,j,k,k'} = -6(\boldsymbol{\alpha}_i'\mathbf{1})(\boldsymbol{\alpha}_j'\mathbf{1})+(-n^2+3n)(\boldsymbol{\alpha}_i'\mathbf{e}_k\boldsymbol{\alpha}_j'\mathbf{e}_k+\boldsymbol{\alpha}_i'\mathbf{e}_{k'}\boldsymbol{\alpha}_j'\mathbf{e}_{k'}+\boldsymbol{\alpha}_i'\mathbf{e}_k\boldsymbol{\alpha}_j'\mathbf{e}_{k'}+\boldsymbol{\alpha}_i'\mathbf{e}_{k'}\boldsymbol{\alpha}_j'\mathbf{e}_k)$$

$$+ 2n\,\boldsymbol{\alpha}_i'\,\Delta_{k,k'}\boldsymbol{\alpha}_j$$

Hence, from (4.14), (4.18) and (4.19), one gets

$$(4.20)\quad \Sigma_\alpha\Big(\sum_{i=1}^{p}\lambda_{i\alpha}\Big)^2 = \Big[\sum_{i=1}^{p}\sum_{j=1}^{p}\boldsymbol{\alpha}_i'((\delta_{i,j,k,k'}))\,\boldsymbol{\alpha}_j\Big]\cdot\frac{b_{2,p}}{n}(n-4)! + \text{a constant term.}$$

To evaluate the bracketted quantity in (4.20), note that

$$(4.21)\quad \boldsymbol{\alpha}_i'((\delta_{i,j,k,k'}))\boldsymbol{\alpha}_j = \sum_{k=1}^{n} a_{ik}a_{jk}\,\delta_{i,j,k,k} + \sum_{k\neq k'=1}^{n} a_{ik}a_{jk'}\,\delta_{i,j,k,k'}$$

$$= \Big[2(n-3)\sum_{k=1}^{n} a_{ik}a_{jk}(\boldsymbol{\alpha}_i'\mathbf{1})(\boldsymbol{\alpha}_j'\mathbf{1})+(n-3)(n^2-2n)\sum_{k=1}^{n} a_{ik}^2a_{jk}^2-n(n-3)\sum_{k=1}^{n}a_{ik}a_{jk}(\boldsymbol{\alpha}_i'\Delta_k\boldsymbol{\alpha}_j)\Big]$$

$$+ \Big[-6(\boldsymbol{\alpha}_i'\mathbf{1})(\boldsymbol{\alpha}_j'\mathbf{1})\sum_{k\neq k'=1}^{n} a_{ik}a_{jk'}+(-n^2+3n)\sum_{k\neq k'=1}^{n}\{a_{ik}a_{jk}+a_{ik}a_{jk'}+a_{ik}a_{jk'}+a_{ik'}a_{jk}\}a_{ik}a_{jk'}$$

$$+ 2n\sum_{k\neq k'=1}^{n} a_{ik}a_{jk'}(\boldsymbol{\alpha}_i'\,\Delta_{k,k'}\boldsymbol{\alpha}_j)\Big]$$

$$= \Big[2(n-3)(\boldsymbol{\alpha}_i'\boldsymbol{\alpha}_j)(\boldsymbol{\alpha}_i'\mathbf{1})(\boldsymbol{\alpha}_j'\mathbf{1})+(n-3)(n^2-2n)\Big(\sum_{k=1}^{n} a_{ik}^2a_{jk}^2\Big)-n(n-3)\Big\{(\boldsymbol{\alpha}_i'\boldsymbol{\alpha}_j)^2+(\boldsymbol{\alpha}_i'\mathbf{1})\sum_{k=1}^{n} a_{jk}^2a_{ik}$$

$$+ (\boldsymbol{\alpha}_j'\mathbf{1})\sum_{k=1}^{n}a_{ik}^2a_{jk}-2\sum_{k=1}^{n} a_{ik}^2a_{jk}^2\Big\}\Big]$$

$$+ \,[-6(\alpha_i'\mathbf{1})(\alpha_j'\mathbf{1})\{(\alpha_i'\mathbf{1})(\alpha_j'\mathbf{1})-\alpha_i'\alpha_j\} + (-n^2+3n)\{(\sum_{k=1}^{n} a_{ik}^2 a_{jk})(\alpha_j'\mathbf{1}) - \sum_{k=1}^{n} a_{ik}^2 a_{jk}^2$$

$$+ (\sum_{k=1}^{n} a_{ik} a_{jk}^2)(\alpha_i'\mathbf{1}) - \sum_{k=1}^{n} a_{ik}^2 a_{jk}^2$$

$$+ (\alpha_i'\alpha_i)(\alpha_j'\alpha_j) - \sum_{k=1}^{n} a_{ik}^2 a_{jk}^2$$

$$+ (\alpha_i'\alpha_j)^2 - \sum_{k=1}^{n} a_{ik}^2 a_{jk}^2\}$$

$$+2n\{3(\alpha_i'\alpha_j)(\alpha_i'\mathbf{1})(\alpha_j'\mathbf{1})-(\alpha_j'\alpha_j)^2-2(\alpha_i'\alpha_i)(\alpha_j'\alpha_j)+(\alpha_i'\alpha_i)(\alpha_j'\mathbf{1})^2+(\alpha_j'\alpha_j)(\alpha_i'\mathbf{1})^2-4(\alpha_j'\mathbf{1})\cdot$$

$$(\sum_{k=1}^{n} a_{ik}^2 a_{jk}) - 4(\alpha_i'\mathbf{1})(\sum_{k=1}^{n} a_{ik} a_{jk}^2) + 4\sum_{k=1}^{n} a_{ik}^2 a_{jk}^2\}$$

$$= 8n(\alpha_i'\alpha_j)(\alpha_i'\mathbf{1})(\alpha_j'\mathbf{1})+(2n^2+4n)(\alpha_i'\alpha_j)^2-(n^2+n)\,(\alpha_i'\alpha_i)(\alpha_j'\alpha_j)+2n(\alpha_i'\alpha_i)(\alpha_j'\mathbf{1})^2$$

$$+2n(\alpha_j'\alpha_j)(\alpha_i'\mathbf{1})^2-2(n^2+n)(\alpha_i'\mathbf{1})(\sum_{k=1}^{n} a_{ik}a_{jk}^2)-2(n^2+n)(\alpha_j'\mathbf{1})(\sum_{k=1}^{n} a_{ik}^2 a_{jk})$$

$$+ (n^3+n^2-4n)(\sum_{k=1}^{n} a_{ik}^2 a_{jk}^2) - 6(\alpha_i'\mathbf{1})^2\,(\alpha_j'\mathbf{1})^2,$$

yielding

$$(4.22)\quad L^*(A) = \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_i'((\sigma_{i,j,k,k'}))\,\alpha_j = 8n\sum_{i=1}^{p}\sum_{j=1}^{p}(\alpha_i'\alpha_j)(\alpha_i'\mathbf{1})(\alpha_j'\mathbf{1})+(-2n^2+4n)(\sum_{i=1}^{p}\sum_{j=1}^{p}(\alpha_i'\alpha_j)^2)$$

$$-\,(n^2+n)\{\sum_{i=1}^{p}(\alpha_i'\alpha_i)\}^2+4n(\sum_{i=1}^{p}\alpha_i'\alpha_i)(\sum_{i=1}^{p}(\alpha_i'\mathbf{1})^2)$$

$$-2(n^2+n)\{\sum_{i=1}^{p}(\alpha_i'\mathbf{1})(\sum_{k'=1}^{n} a_{ik'}(\sum_{j=1}^{p} a_{jk'}^2))\}$$

$$-2(n^2+n)\{\sum_{j=1}^{p}(\alpha_j'\mathbf{1})(\sum_{k'=1}^{n} a_{jk'}(\sum_{i=1}^{p} a_{ik'}^2))\}$$

$$+(n^3+n^2-4n)\sum_{k=1}^{n}(\sum_{i=1}^{p} a_{ik}^2)^2-6\{\sum_{i=1}^{p}(\alpha_i'\mathbf{1})^2\}^2.$$

This completes the proof of the lemma. □

Remark 4.1. When $p = 1$, $b_{2,p}$ boils down to the univariate coefficient of kurtosis and $L^*(A)$ to a multiple of $K_4(a)$, the fourth k-statistic, as defined in Ferguson (1961).

Remark 4.2. It may be noted that the optimality result in Theorem 4.1, stated in terms of $L^*(A)$, is conditional on a specified matrix A. As in Ferguson (1961) and Schwager and Margolin (1982), we notice that whenever $\underset{\sim}{1}'A = \underset{\sim}{0}$ and the number M of outliers (which is at least p by our assumption made in this Section) satisfies $p < M \leq \frac{n}{3}$, $L^*(A) > 0$ whatever A, thereby proving a sort of universal optimality of the $b_{2,p}$-test. This is because when $\underset{\sim}{\alpha}_i'\underset{\sim}{1} = 0$, $\forall_i$, $L^*(A)$ as defined in (4.22) above reduces to

$$L^*(A) = (n^3+n^2-4n)\sum_{k=1}^{M}\left(\sum_{i=1}^{p} a_{ik}^2\right)^2 - (2n^2-4n)\sum_{i=1}^{p}\sum_{j=1}^{p}(\underset{\sim}{\alpha}_i'\underset{\sim}{\alpha}_j)^2-(n^2+n)\left\{\sum_{i=1}^{p}(\alpha_i'\alpha_j)\right\}^2$$

$$\geq (n^3+n^2-4n)\sum_{k=1}^{M}\left(\sum_{i=1}^{p} a_{ik}^2\right)^2-3(n^2-n)\left\{\sum_{i=1}^{p}(\alpha_i'\alpha_i)\right\}^2, \text{ by Cauchy-Schwartz inequality}$$

$$\geq \{(n^3+n^2-4n)-3M(n^2-n)\}\left\{\sum_{k=1}^{M}\left(\sum_{i=1}^{p} a_{ik}^2\right)^2\right\} > 0 \quad \text{if} \quad M \leq \frac{n}{3}.$$

For an arbitrary A, the expression $L^*(A)$ can perhaps be further simplified into a more illuminating form comparable to L(A) defined in (3.11).

Remark 4.3. As mentioned in Section 1, the result of this paper can be viewed as an inference robustness property of Mardia's $b_{2,p}$-test against a class of non-normal distributions. A more general formulation of the problem would be to consider the class of np-dimensional left $\mathcal{O}(n)$-invariant distributions satisfying

$L(QX) = L(X)$ for all $Q \in O(n)$. The density of such a distribution can be expressed as $f(X) = \psi(X'X)$ where ψ: $S(p) \to [0,\infty)$. A left $O(n)$-invariant distribution which is not elliptically symmetric is the matrix variate t-distribution, whose density is given by

$$f_0(X) = c|I_p + X'X|^{-(m+n)/2}$$

where $m > 0$ and c is a normalizing constant. While similar results can be obtained working with the density f_0, for an arbitrary ψ the treatment seems to be difficult. This will be taken up in a future investigation.

APPENDIX

Here we provide proofs of (4.16) and (4.17). Towards this end, note that if z is any $n \times 1$ vector such that $z'1 = 0$ and z_α denotes a permutation of the coordiates of z, then straightforward arguments and calculations establish that

$$
(A.1) \quad \Sigma_\alpha z_\alpha z_\alpha' e_1 e_1' z_\alpha z_\alpha' = \begin{bmatrix}
(n-1)!\sum_1^n z_i^4 & (n-2)!\sum\limits_i\sum\limits_{\ell\neq i} z_i^3 z_\ell & (n-2)!\sum\limits_i\sum\limits_{\ell\neq i} z_i^3 z_\ell & \cdots\cdots & (n-2)!\sum\limits_i\sum\limits_{\ell\neq i} z_i^3 z_\ell \\
\cdots & (n-2)!\sum\limits_{i\neq j} z_i^2 z_j^2 & (n-3)!\sum\limits_{\substack{i\,j\,j' \\ j\neq j'(\neq i)}}\sum\sum z_i^2 z_j z_{j'} & \cdots & (n-3)!\sum\limits_{\substack{i\,j\,j' \\ j\neq j'(\neq i)}}\sum\sum z_i^2 z_j z_{j'} \\
\cdots & \cdots\cdots & (n-2)!\sum\limits_{i\neq j} z_i^2 z_j^2 & \cdots & (n-3)!\sum\limits_{\substack{i\,j\,j' \\ j\neq j'(\neq i)}}\sum\sum z_i^2 z_j z_{j'} \\
 & & & \ddots & \vdots \\
\cdots & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & (n-2)!\sum\limits_{i\neq j} z_i^2 z_j^2
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(n-1)!\Delta & -(n-2)!\Delta & -(n-2)!\Delta & \cdots\cdots & -(n-2)!\Delta \\
\cdots & (n-2)!(c^2-\Delta) & (n-3)!(2\Delta-c^2) & \cdots & (n-3)!(2\Delta-c^2) \\
\cdots & \cdots\cdots\cdots & (n-2)!(c^2-\Delta) & \cdots & (n-3)!(2\Delta-c^2) \\
 & & & \ddots & \vdots \\
\cdots & \cdots\cdots & \cdots\cdots\cdots & & (n-2)!)c^2-\Delta)
\end{bmatrix}
$$

and in general $\Sigma_\alpha z_\alpha z_\alpha' e_i e_i' z_\alpha z_\alpha'$ is obtained by interchanging the first row and first column of (A.1) with the ith row and ith column; and

$$\text{(A.2)}\quad \Sigma_{\alpha} z_{\alpha} z_{\alpha}' e_1 e_2' z_{\alpha} z_{\alpha}' = \left[\begin{array}{ccccc}
(n-2)!\sum\sum_{i\neq j} z_i^3 z_j & (n-2)!\sum\sum_{i\neq j} z_i^2 z_j^2 & (n-3)!\sum\sum\sum_{i\neq j\neq \ell} z_i^2 z_j z_\ell & \cdots & (n-3)!\sum\sum\sum_{i\neq j\neq \ell} z_i^2 z_j z_\ell \\
\cdots & (n-2)!\sum\sum_{i\neq j} z_i^3 z_j & (n-3)!\sum\sum\sum_{i\neq j\neq \ell} z_i^2 z_j z_\ell & \cdots & (n-3)!\sum\sum\sum_{i\neq j\neq \ell} z_i^2 z_j z_\ell \\
\cdots & \cdots & (n-3)!\sum\sum\sum_{i\neq j\neq \ell} z_i^2 z_j z_\ell & \cdots & (n-4)!\sum\sum\sum\sum_{i\neq j\neq \ell\neq \ell'} z_i z_j z_\ell z_{\ell'} \\
 & & & (n-3)!\sum\sum\sum_{i\neq j\neq \ell} z_i^2 z_j z_\ell & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & (n-3)!\sum\sum\sum_{i\neq j\neq \ell} z_i^2 z_j z_\ell
\end{array}\right.$$

$$= \left[\begin{array}{ccccc}
-(n-2)!\Delta & (n-2)!(c^2-\Delta) & (n-3)!(2\Delta-c^2) & (n-3)!(2\Delta-c^2) \cdots & (n-3)!(2\Delta-c^2) \\
\cdots & -(n-2)!\Delta & (n-3)!(2\Delta-c^2) & (n-3)!(2\Delta-c^2) \cdots & (n-3)!(2\Delta-c^2) \\
\cdots & \cdots & (n-3)!(2\Delta-c^2) & (n-4)!(3c^2-6\Delta) \cdots & (n-4)!(3c^2-6\Delta) \\
\cdots & \cdots & \cdots & (n-3)!(2\Delta-c^2) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & (n-3)!(2\Delta-c^2)
\end{array}\right.$$

and in general $\Sigma_{\alpha} z_{\alpha} z_{\alpha}' e_i e_j' z_{\alpha} z_{\alpha}'$ is obtained by interchanging the first row, first column of (A.2) with the ith row, ith column and second row, second column of (A.2) with the jth row, jth column, where $\Delta = \sum_1^n z_i^4$ and $c = \sum_1^n z_i^2$. In order to evaluate (4.16) and (4.17) we need an extension of these results requiring computations of the following quantities.

$$\text{(A.3)}\quad \text{(i)} \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{i=1}^{n} z_{i\ell}^2 z_{i\ell'}^2 = \sum_{i=1}^{n}\left(\sum_{\ell=1}^{p} z_{i\ell}^2\right)\left(\sum_{\ell'=1}^{p} z_{i\ell'}^2\right) = \sum_{i=1}^{n}\left(\sum_{\ell=1}^{p} z_{i\ell}^2\right)^2$$

$$= \sum_{i=1}^{n}\left[\sum_{\ell=1}^{n}\{e_i'(x-1\bar{x}')\frac{p_\ell p_\ell'}{\lambda_\ell}(x-1\bar{x}')'e_i\}\right]^2 = \sum_{i=1}^{n}\{e_i'(x-1\bar{x}')S^{-1}(x-1\bar{x}')'e_i\}^2$$

$$= \sum_{i=1}^{n} \{(\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})' S^{-1} (\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})\}^2 = \frac{b_{2,p}}{n} \ ;$$

(ii) $$\sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i \neq}^{n} \sum_{i'=1}^{n} z_{i\ell}^2 z_{i'\ell'} z_{i\ell'} = \sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i=1}^{n} \sum_{i'=1}^{n} z_{i\ell}^2 z_{i\ell'} z_{i'\ell'} - \sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i=1}^{n} z_{i\ell}^2 z_{i\ell'}^2$$

$$= -\frac{b_{2,p}}{n} \text{ by (i) and } \sum_{i'=1}^{n} z_{i'\ell'} = 0,\ \forall\ \ell' \ ;$$

(iii) $$\sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i \neq}^{n} \sum_{i'=1}^{n} z_{i\ell}^2 z_{i'\ell'}^2 = \sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i=1}^{n} \sum_{i'=1}^{n} z_{i\ell}^2 z_{i'\ell'}^2 - \sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i=1}^{n} z_{i\ell}^2 z_{i\ell'}^2 = \{\sum_{\ell=1}^{p} (\sum_{i=1}^{n} z_{i\ell}^2)\}^2 - \frac{b_{2,p}}{n}$$

$$= \{\sum_{\ell=1}^{p} \frac{P_\ell' S P_\ell}{\lambda_\ell}\}^2 - \frac{b_{2,p}}{n} = p^2 - \frac{b_{2,p}}{n} \ ;$$

(iv) $$\sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i \neq}^{n} \sum_{i' \neq}^{n} \sum_{i''=1}^{n} z_{i\ell} z_{i\ell'} z_{i'\ell} z_{i''\ell'} = -\sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i \neq}^{n} \sum_{i'=1}^{n} z_{i\ell} z_{i\ell'} z_{i'\ell} (z_{i\ell'} + z_{i'\ell'})$$

$$= -\sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i \neq i'=1}^{n} z_{i\ell'}^2 z_{i\ell} z_{i'\ell} - \sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i \neq i'=1}^{n} z_{i\ell} z_{i\ell'} z_{i'\ell} z_{i'\ell'}$$

$$= \frac{b_{2,p}}{n} - \sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i=1}^{n} \sum_{i'=1}^{n} z_{i\ell} z_{i\ell'} z_{i'\ell} z_{i'\ell'} + \sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{i=1}^{n} z_{i\ell}^2 z_{i\ell'}^2$$

$$= 2(\frac{b_{2,p}}{n}) - \sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} (\sum_{i=1}^{n} z_{i\ell} z_{i\ell'})^2 = 2(\frac{b_{2,p}}{n}) - \sum_{i=1}^{n} \sum_{i'=1}^{n} \{\sum_{\ell=1}^{p} \underset{\sim}{e}_i' (\underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}') \frac{P_\ell P_\ell'}{\lambda_\ell} (\underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')' \underset{\sim}{e}_{i'}\}^2$$

$$= 2(\frac{b_{2,p}}{n}) - \mathrm{tr}[(\underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}') S^{-1} (\underset{\sim}{x} - \underset{\sim}{1}\bar{\underset{\sim}{x}}')']^2 = 2(\frac{b_{2,p}}{n}) - p, \text{ using (i), (ii),}$$

the definition of $S^{-1} = \sum_{\ell=1}^{p} P_\ell P_\ell' / \lambda_\ell$ and the fact that for any symmetric matrix B, $\sum_i \sum_{i'} (\underset{\sim}{e}_i' B \underset{\sim}{e}_{i'})^2 = \mathrm{tr}\, B^2$;

$$\text{(v)} \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{i\neq i'\neq i''\neq i'''=1}^{n} z_{i\ell}z_{i'\ell}z_{i''\ell'}z_{i'''\ell'} = -\sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{i\neq i'\neq i''=1}^{n} z_{i\ell}z_{i'\ell}z_{i''\ell'}$$

$$(z_{i\ell'}+z_{i'\ell'}+z_{i''\ell'}) = -4(\frac{b_{2,p}}{n}) + 2p - \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{i\neq i'\neq i''=1}^{n} z^2_{i''\ell'}z_{i\ell}z_{i'\ell}$$

$$= -4(\frac{b_{2,p}}{n}) + 2p - \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{i\neq i'=1}^{n} z_{i\ell}z_{i'\ell}\{\sum_{i=1}^{n} z^2_{i\ell'} - z^2_{i\ell'} - z^2_{i'\ell'}\}$$

$$= -4(\frac{b_{2,p}}{n}) + 2p - \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{i\neq i'=1}^{n} z_{i\ell}z_{i'\ell} - 2(\frac{b_{2,p}}{n})\text{[using } \sum_{i=1}^{n} z^2_{i\ell'}=1,\ \forall\ \ell' \text{ and (ii)]}$$

$$= -6(\frac{b_{2,p}}{n}) + 2p + p\sum_{\ell=1}^{p}\sum_{i=1}^{n} z^2_{i\ell} \text{ [using } \sum_{i'\neq i=1}^{n} z_{i'\ell} = -z_{i\ell},\ \forall\ \ell]$$

$$= -6(\frac{b_{2,p}}{n}) + 2p + p^2 .$$

In (A.3) above, we have taken $z_{\ell} = (x-1\bar{x}')P_{\ell}/\sqrt{\lambda_{\ell}}$, $\ell = 1,\ldots,p$ and $z_{i\ell}$ stands for the ith element of the vector z_{ℓ}, $i = 1,\ldots,n$. An argument similar to that used to proved (A.1) and (A.2) and the use of the computations in (A.3) lead to the final expressions.

$$\text{(A.4)} \sum_{\ell=1}^{p}\sum_{\ell'=1}^{p}\sum_{\alpha} z_{\ell\alpha}z'_{\ell\alpha}e_1e'_1z_{\ell'\alpha}z'_{\ell'\alpha} =$$

$$\left(\begin{array}{ccccc}
(n-1)!(\frac{b_{2,p}}{n}) & -(n-2)!(\frac{b_{2,p}}{n}) & -(n-2)!(\frac{b_{2,p}}{n}) & \ldots & -(n-2)!(\frac{b_{2,p}}{n}) \\
\ldots & (n-2)!(p^2-\frac{b_{2,p}}{n}) & (n-3)!(2\frac{b_{2,p}}{n}-p) & \ldots & (n-3)!(2\frac{b_{2,p}}{n}-p) \\
\ldots & \ldots & (n-2)!(p^2-\frac{b_{2,p}}{n}) & \ldots & (n-3)!(2\frac{b_{2,p}}{n}-p) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots\ \ldots & \ldots & \ldots & (n-2)!(p^2-\frac{b_{2,p}}{n})
\end{array}\right.$$

$$= \left(\frac{b_{2,p}}{n}\right)(n-3)! \begin{pmatrix} (n-1)(n-2) & -(n-2) & -(n-2) & \dots & -(n-2) \\ & -(n-2) & 2 & \dots & 2 \\ & & -(n-2) & \dots & 2 \\ & & & & -(n-2) \end{pmatrix}$$

$$+ p(n-3)! \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ & p(n-2) & -1 & \dots & -1 \\ & & p(n-2) & \dots & -1 \\ & & & \ddots & \vdots \\ & & & & p(n-2) \end{pmatrix}$$

and in general $\sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{\alpha} z_{\ell\alpha} z'_{\ell\alpha} e_k e'_k z_{\ell'\alpha} z'_{\ell'\alpha}$ is obtained by interchanging the first row, first column of (A.4) with the kth row, kth column, thus proving (4.16);

(A.5) $\sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{\alpha} z_{\ell\alpha} z'_{\ell\alpha} e_1 e'_2 z_{\ell'\alpha} z'_{\ell'\alpha} =$

$$\begin{pmatrix} -(n-2)!\frac{b_{2,p}}{n} & (n-2)!(p^2-\frac{b_{2,p}}{n}) & (n-3)!(\frac{2b_{2,p}}{n}-p) & (n-3)!(\frac{2b_{2,p}}{n}-p) & \dots & (n-3)!(\frac{2b_{2,p}}{n}-p) \\ & -(n-2)!\frac{b_{2,p}}{n} & (n-3)!(\frac{2b_{2,p}}{n}-p) & (n-3)!(\frac{2b_{2,p}}{n}-p) & \dots & (n-3)!(\frac{2b_{2,p}}{n}-p) \\ & & (n-3)!(\frac{2b_{2,p}}{n}-p) & (n-4)!(-\frac{6b_{2,p}}{n}+2p+p^2) & \dots & (n-4)!(-\frac{6b_{2,p}}{n}+2p+p \\ & & & (n-3)!(\frac{2b_{2,p}}{n}-p) & \dots & (n-4)!(-6\frac{b_{2,p}}{n}+2p+p^2) \\ & & & & & (n-3)!(\frac{2b_{2,p}}{n}-p) \end{pmatrix}$$

$$= \frac{b_{2,p}}{n}(n-4)! \begin{pmatrix} -(n-2)(n-3) & -(n-2)(n-3) & 2(n-3) & 2(n-3) & \cdots & 2(n-3) \\ & -(n-2)(n-3) & 2(n-3) & 2(n-3) & \cdots & 2(n-3) \\ & & 2(n-3) & -6 & \cdots & -6 \\ & & & (2(n-3) & \cdots & -6 \\ & & & & \ddots & \vdots \\ & & & & & 2(n-3) \end{pmatrix}$$

$$+P(n-4)! \begin{pmatrix} 0 & p(n-2)(n-3) & -(n-3) & -(n-3) & \cdots & -(n-3) \\ & 0 & -(n-3) & -(n-3) & \cdots & -(n-3) \\ & & -(n-3) & (p+2) & \cdots & (p+2) \\ & & & -(n-3) & \cdots & (p+2) \\ & & & & \ddots & \vdots \\ & & & & & -(n-3) \end{pmatrix}$$

and in general $\sum_{\ell=1}^{p} \sum_{\ell'=1}^{p} \sum_{\alpha} \underline{z}_{\ell\alpha} \underline{z}'_{\ell\alpha} \underline{e}_k \underline{e}'_{k'} \underline{z}_{\ell'\alpha} \underline{z}'_{\ell'\alpha}$ is obtained by interchanging the first row, first column of (A.5) with the kth row, kth column and second row, second column of (A.5) with the k'th row, k'th column, thus establishing (4.17).

REFERENCES

[1] Barnett, V. and Lewis, T. (1978). Outliers in Statistical Data. Wiley, New York.

[2] Chikkagoudar, M.S. and Kunchur, S.H. (1980). Estimation of the mean of an exponential distribution in the presence of an outlier. Canadian Journal of Statistics, 8, 59-63.

[3] Dawid, A.P. (1977). Sperical matrix distributions and a multivariate model. J. Roy. Statist. Soc., Ser. B, 39, 254-261.

[4] Eaton, M.L. and Kariya, T. (1975). Tests on means with additional information. Technical Report No. 243, University of Minnesota.

[5] Ferguson, T.S. (1961). On the rejection of outliers. Proc. Fourth Berkeley Symposium Math. Statist. Prob. I, 253-287.

[6] Gnanadesikan, R. (1977). Methods for Statistical Data Analysis of Multivariate Observations. Wiley, New York.

[7] Hawkins, D.M. (1980). Identification of Outliers. Chapman and Hall, New York.

[8] Healy, M.J.R. (1968). Multivariate normal plotting. Appl. Statist. 17, 157-161.

[9] Joshi, P.C. (1972). Efficient estimation of the mean of an exponential distribution when an outlier is present. Technometrics 14, 137-143.

[10] Kale, B.K. and Sinha, S.K. (1971). Estimation of expected life in the presence of an outlier observation. Technometrics 13, 755-759.

[11] Karlin, S. and Truax, D.R. (1960). Slippage problems. Ann. Math. Statist. 31 , 296-324.

[12] Kariya, T. and Eaton, M.L. (1977). Robust tests for spherical symmetry. Ann. Statist. 5, 206-215.

[13] Kariya, T. (1978). The general MANOVA problem. Ann. Statist. 6, 200-214.

[14] Kariya, T. (1981a). A robustness property of Hotelling's T^2-Test. Ann. Statist., 9, 211-214.

[15] Kariya, T. (1981b). Tests for the independence between two seemingly unrelated regression equations. Ann. Statist. 9, 381-390.

[16] Mardia, K.V. (1970). Measures of multivariate skewness and kurtosis with applications. Biometrika, 57, 519-530.

[17] Rohlf, F.J. (1975). Generalization of the gap test for the detection of multivariate outliers. Biometrics 31, 93-101.

[18] Schwager, S.J. and Margolin, B.H. (1982). Detection of multivariate normal outliers. Ann. Statist., 10, 943-954.

[19] Sinha, S.K. (1972). Reliability estimation in life testing in the presence of an outlier observation. Operations Research 20, 888-894.

[20] Sinha, S.K. (1973a). Estimation of the parameters of a two-parameter exponential distribution when an outlier may be present. Utilitas Mathematica 3, 75-82.

[21] Sinha, S.K. (1973b). Distributions of order statistics and estimation of mean life when an outlier may be present. The Canadian Journal of Statistics 1, 119-121.

[22] Sinha, S.K. (1973c). Life testing and reliability estimation for non-homogeneous data - a Bayesian approach. Comm. Statist 2, 235-243.

[23] Sinha, S.K. (1975). Some distributions relevant in life testing when an outlier may be present. Sankhyā B, 37, 100-105.

[24] Sinha, S.K. (1976). On the estimation of the exponential lower bound when an outlier may be present. Journal of Indian Society of Agricultural Statistics, 28, 15-18.

[25] Sinha, B.K. and Drygas, H. (1982). Robustness porperties of the F-test and best linear unbiased estimators in linear models. Tech. Report No. 82-28, Center for Multivariate Analysis, University of Pittsburgh.

[26] Sinha, B.K., Sarkar, S.K. and Krishnaiah, P.R. (1982). Tests of parameters of a bivariate normal population with unbalanced data. To appear in Ann. Inst. Statist. Math.

[27] Siotani, M. (1959). The extreme value of the generalized distances of the individual points in the multivariate normal sample. Ann. Inst. Statist. Math. Tokyo 10, 183-208.

[28] Wijsman, R.A. (1967). Cross-sections of orbits and their application to densities of maximal invariants. Fifth Berkeley Symp. Math. Statist. Prob. 1, 389-400.

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An extension fo Ferguson's [Fourth Berkeley Symposium on Probability and Mathematical Statistics, 1961, Volume 1] univariate normal results for rejection of outliers is made to the multivariate case with mean slippage. The formulation is more general than that in Schwager and Margolin [Ann. Statist., 1982, Vol. 10, No. 3, 943-954] and the approach is also different. The main result can be viewed as a robustness property of Mardia's locally optimum multivariate normal kurtosis test to detect outliers against nonnormal multivariate distributions.

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